third harmonic are needed to account for the behaviour of the vertical velocity component in the midplan of the cell. We have searched for the presence of the fourth and fifth harmonics, but could not establish their presence. We checked also that there was no significant velocity component along $y'y$ i.e. the velocity field was bidimensional. The variations of $\tilde{W}^1$ and $\tilde{W}^3$ with respect to a are given on Fig. 3 and Table 2, for a value of $Ra \simeq 11\,400$. We see that the amplitude of the fundamental mode $\tilde{W}^1$ increases with a, when $\tilde{W}^3$ is decreasing; so the amount of anharmonicity is greater when the wavelength is smaller* (see Fig. 2). From an other point of view, we checked the variation of $\tilde{W}_1$ and $\tilde{W}_3$ with Ra for the structure with $a = 2.57$ and found approximatively the expected power law dependences.

On the Fig. 3 (and Table 2), we give also the velocity amplitudes deduced from the calculated values $W^{(1)}$ and $W^{(3,3)}$. These ones are represented by striped areas, their width being given by the experimental uncertainty on D and Ra. Furthermore, we draw for comparison the values deduced from Busse's results using a Galerkin procedure. The corresponding amplitudes† are not precise for they are obtained from an estimate of parameters taken from a published diagram (uncertainty $\pm 5\%$).

In conclusion, the dependence with respect to a of the measured amplitude of the fundamental mode is well described by the perturbative method. The higher values deduced from Busse's calculations can be mainly explained by the fact that his parameter which gives the fundamental amplitude does not follow the power law in $[(Ra - Ra_c)/Ra_c]^{1/2}$. On the contrary, for the third harmonic amplitudes, the agreement between the experimental points and the values calculated from Busse is very good when the perturbative method gives lower values and a different variation law with a.

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*We can notice that the total mass flux carried by convection and deduced from velocity profiles is decreasing when Λ increases.

†There is a misprint in Busse's paper [3] since the values of the coefficients $b_{\lambda\nu}$ reported in its Fig. 3 are not coherent with the results of the Table 1. They become consistent if we drop a factor π in the value of $b_{\lambda\nu}$.

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A LOCAL SIMILARITY MODEL FOR THE HEAT FLUX EQUATION IN A TURBULENT BOUNDARY LAYER

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NOMENCLATURE

a_1, a_2, a_3, a_θ, turbulence structure parameters, assumed constant;
L, length scale defined by equation (2);
l, mixing length $= -\overline{uv}^{1/2}/(\partial U/\partial y)$;
Pr_t, turbulent Prandtl number;
p, turbulent kinematic pressure fluctuation;
q^2, turbulent kinetic energy $(=\overline{u^2}+\overline{v^2}+\overline{w^2})$;
T, difference between local and mean and free stream ambient temperatures;
U, local mean velocity;
u, v, w, velocity fluctuations in x, y and z directions;
$-\overline{uv}$, Reynolds shear stress;
$-\overline{u\theta}$, longitudinal heat flux;
u_τ, friction velocity;
$\overline{v\theta}$, normal heat flux;
x, y, z, coordinates in longitudinal (streamwise), normal (to wall) and spanwise directions respectively;

x_i, spatial coordinates, identical to x, y and z when $i = 1, 2, 3$ respectively.

Greek symbols

α, thermal diffusivity;
β, constant defined by equation (8);
δ, momentum boundary layer thickness (99.5% of free stream velocity);
δ_T, thermal layer thickness (99.5% of difference between wall and free stream temperature);
θ, temperature fluctuation;
θ_τ, friction temperature, equal to ratio of thermometric wall heat flux to friction velocity;
ν, kinematic viscosity.

EMPIRICAL models of transport equations for the turbulent heat flux are currently being used in calculation methods for both laboratory and planetary turbulent boundary layers [1–5]. In the absence of buoyant generation, the differential equation for $\overline{\theta u_i}$ may be written [6] as

$$\underbrace{U_k \frac{\partial}{\partial x_k}\overline{\theta u_i}}_{\textit{advection}} + \underbrace{\overline{u_i u_j}\frac{\partial T}{\partial x_j} + \overline{\theta u_k}\frac{\partial U_i}{\partial x_k}}_{\textit{production}} + \underbrace{\frac{\partial}{\partial x_m}(\overline{u_m \theta u_i})}_{\textit{diffusion}}$$

$$+\,\overline{\theta\frac{\partial p}{\partial x_i}} - \underbrace{\alpha\overline{u_i \nabla^2 \theta} - \nu\overline{\theta \nabla^2 u_i}}_{\text{dissipation}} = 0 \quad (1)$$

The last two terms may be interpreted as dissipation terms and are expected to be small by virtue of isotropy of the fine scale motion. When the flow is approximately self-preserving, advection and diffusion terms are expected to be small in comparison with the production term so that the temperature–pressure gradient correlation would then represent the major destruction or sink term for $\overline{\theta u_i}$. We do not yet have an explanation for the physical processes that are represented by the temperature–pressure gradient correlation term but it is obviously important to model this term accurately if equation (1) is used in a calculation scheme. The simplest model is of a "return to isotropy" form, which expresses a tendency for the pressure fluctuations to reduce any correlation between θ and u_i, and is represented by

$$\overline{\theta\frac{\partial p}{\partial x_i}} = \frac{\overline{q^2}^{1/2}}{L}\overline{\theta u_i}, \quad (2)$$

where only turbulence terms are included. More general proposals, reviewed in [7], include mean strain and buoyancy effects in the RHS of equation (2). The length scale L is expected to be proportional to a length scale representative of the energy containing eddies. Donaldson [1] chose a value of L ($\simeq 0.15\delta$ in the outer layer) by optimising agreement of his method with boundary-layer results. In [2] and [3], $\overline{\theta \partial p/\partial x_i}$ is assumed to be proportional to $\overline{\theta u_i}/\tau$, where τ is an integral time scale (set by the model) and where the proportionality constant is determined by forcing the $\overline{\theta v}$ equation to satisfy observations in the neutral surface layer of the atmosphere. In the present note, $\overline{\theta \partial p/\partial x_i}$ is estimated from an experimental budget of $\overline{\theta u_i}$ in an approximately self-preserving thermal boundary layer and the resulting values of L are compared with those chosen by Donaldson [1] and others.

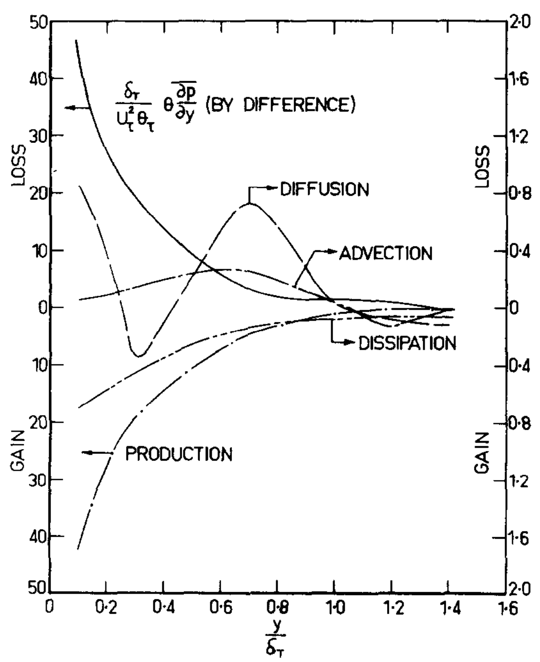

FIG. 1. Budget of $\overline{v\theta}$ in heated part of the flow at $x/\delta_0 = 42.9$.

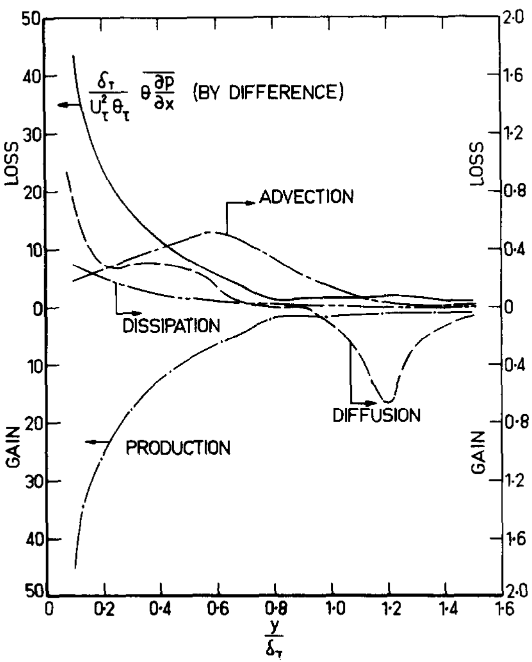

FIG. 2. Budget of $\overline{u\theta}$ in heated part of the flow at $x/\delta_0 = 42.9$.

Budgets of heat fluxes $\overline{v\theta}$ (in direction normal to the wall) and $\overline{u\theta}$ (longitudinal direction) have been measured [8, 9] in a thermal layer which was obtained by subjecting a fully developed boundary layer to a step change in surface heat flux. Upstream of the step the surface heat flux was zero while downstream of the step, the magnitude of the constant heat flux is small enough for temperature to be considered as a passive contaminant of the flow. At the step, the velocity boundary layer is self-preserving with a thickness δ_0 of 4.5 cm and a Reynolds number Re_θ ($U_1\theta_0/\nu$) of approximately 3000 ($U_1 \simeq 9.45$ m/s). Fluctuations u, v and θ were obtained with a combined X-wire/single wire probe arrangement. A miniature DISA X-wire (5 μm dia. Pt coated tungsten wire) operated by two channel of DISA 55M01 constant temperature anemometers was used to measure u and v. The temperature fluctuation θ was measured with a 1 μm dia. platinum cold wire at a distance of about 1 mm from the geometrical centre of the X-wire. This cold wire was operated by a constant current anemometer with the value of the current set at 0.2 mA. Contamination of the X-wire signals by θ was removed using the method described in [10]. Budgets of $\overline{v\theta}$ and $\overline{u\theta}$ shown in Figs. 1 and 2 respectively were obtained at a distance of $42.9\delta_0$ downstream of the step, where the thermal layer is nearly self-preserving but with a thickness δ_T equal to only 66% of the local velocity boundary-layer thickness δ ($\simeq 8.64$ cm). In Figs. 1 and 2 all terms in equation (1) were normalized by δ_T, the friction velocity u_τ and the friction temperature θ_τ while the averaging was performed only during those periods for which the flow was heated. The results of Figs. 1 and 2 clearly indicate that the temperature–pressure gradient correlation effectively counteracts the production term. All other terms in equation (1) are one order of magnitude smaller than the two major terms. At very small distances from the step budgets of $\overline{u\theta}$ and $\overline{v\theta}$ given in [8] show that the diffusion term may become comparable with the temperature–pressure gradient correlation.

Equation (1) may be approximated, for $\overline{v\theta}$ in a boundary layer with negligible advection and dissipation, by

$$\overline{v^2}\frac{\partial T}{\partial y} = -\frac{\overline{q^2}^{1/2}}{L}\overline{v\theta}. \tag{3}$$

With the assumption that $\overline{v^2} = a_2(-\overline{uv})$, with a_2 assumed constant, equation (3) may be rewritten, after some manipulation, as

$$\frac{L}{l} = \frac{1}{a_1^{1/2}a_2 Pr_t} \tag{4}$$

where $l[\equiv -\overline{uv}^{1/2}(\partial U/\partial y)^{-1}]$ is the mixing length, Pr_t is the turbulent Prandtl number $\overline{uv}(\partial T/\partial y)/\overline{v\theta}(\partial U/\partial y)$ and a_1 is the parameter $-\overline{uv}/\overline{q^2}$. With $a_1 \simeq 0.15$ [11], $a_2 \simeq 1.5$ [7] and $Pr_t \simeq 0.9$ [12, 13], L is approximately equal to twice the mixing length. In the case of $\overline{u\theta}$, equation (1) may be approximated by

$$\overline{v\theta}\frac{\partial U}{\partial y} + \overline{uv}\frac{\partial T}{\partial y} = -\frac{\overline{q^2}^{1/2}}{L}\overline{\theta u}. \tag{5}$$

With the assumption that $-\overline{u\theta} = a_3\overline{v\theta}$, equation (5) may be recast as

$$\frac{L}{l} = \frac{a_3}{a_1^{1/2}}(1+Pr_t)^{-1}. \tag{6}$$

With $a_3 = 1.5$, a value suggested by the measurements in [13] and [14], equation (6) yields $L \simeq 2l$, in agreement with the result obtained from equation (3). Launder [7] has made a comprehensive review of the proposed values for the coefficient c_{1c} in the model

$$\overline{p\frac{\partial\theta}{\partial x_i}} = -c_{1c}\frac{2\varepsilon}{\overline{q^2}}\overline{u_i\theta} \tag{7}$$

where ε is the dissipation of turbulent energy. Assuming that $\overline{p\partial\theta/\partial x_i} \simeq -\overline{\theta\partial p/\partial x_i}$, (7) may be expressed as

$$\overline{\theta\frac{\partial p}{\partial x_i}} = 2c_{1c}a_1^{3/2}\frac{\overline{q^2}^{1/2}}{L_\varepsilon}\overline{u_i\theta} \tag{8}$$

where L_ε is the dissipation length scale $(-\overline{uv})^{3/2}/\varepsilon$. With $L_\varepsilon \simeq l$ [11], equation (8) yields $L \simeq 2.5l$ when the value of c_{1c} ($\simeq 3.4$), (as recommended by Launder [7]) is used.

It is worth noting that the turbulent Prandtl number Pr_t may also be written as

$$Pr_t = \frac{a_1^{1/2}B}{a_\theta} \tag{9}$$

where the turbulence structure parameter a_θ, introduced in [15], is defined as

$$a_\theta = \frac{\overline{v\theta}}{\overline{\theta^2}^{1/2}(-\overline{uv})^{1/2}}$$

and B is the non-dimensional ratio

$$B = -\frac{\overline{q^2}^{1/2}\partial T/\partial y}{\overline{\theta^2}^{1/2}\partial U/\partial y}$$

introduced in [13, 16] and found to be equal to 1.5 over the major part of the layer. With $a_\theta = 0.64$, as given by results in [13] and [14], equation (9) yields $Pr_t \simeq 0.9$, which is consistent with our chosen value of Pr_t. It should also be noted that if the effect of the mean strain rate is included in equation (2), in the form [17, 4]

$$\overline{\theta\frac{\partial p}{\partial x_i}} = \frac{\overline{q^2}^{1/2}}{L}\overline{\theta u_i} + \beta\frac{\partial U_i}{\partial x_j}\overline{\theta u_j} \tag{10}$$

equation (6) is modified to

$$\frac{L}{l} = \frac{a_3}{a_1^{1/2}}(1+\beta+Pr_t)^{-1}. \tag{11}$$

With $\beta = -0.5$, the value used in [17], equation (11) yields a value of the ratio L/l which is 38% higher than that given by equation (6). Launder [7] has already noted that the value of c_{1c} does not depend significantly on the inclusion of the mean strain rate effect in equation (2).

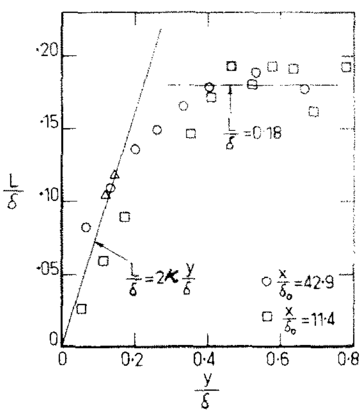

FIG. 3. Distribution of length scale L, defined in equation (2).

It should finally be mentioned that the validity of the result $L \simeq 2l$ ought to extend over the major part of the boundary layer since the turbulence structure parameters a_1, a_2, a_3, a_θ and the non-dimensional ratios Pr_t or B are approximately constant except in the region very near the wall and near the outer edge of the layer. Values of L (Fig. 3), derived from the experimental budget of $\overline{u\theta}$ for the streamwise stations of the thermal layer, are in fair agreement with a line of slope 2κ (κ is the von Karman constant) in the logarithmic region of the layer and with a constant value of 0.18 in the region $0.3 < y/\delta < 0.8$. This value of 0.18 is approximately twice the value of the mixing length l reported by Bradshaw [11] in the same region of the layer.

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